MUTUAL EFFECT OF LONG WAVES AND TURBULENCE ON THE SURFACE OF A LIQUID

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UDC 532.517

The interaction of a long coherent wave with the turbulence on the surface of a liquid is investigated within the framework of the theory of weak turbulence. A closed system of equations is obtained which consists of the dynamic equation for the coherent wave and equations of kinetic type describing the turbulent subsystem. It is shown that because of the interaction with the turbulent subsystem, coherent waves with wave vectors identical in magnitude but opposite in direction are coupled. The additional attenuation of the coherent wave because of the interaction is estimated; this attenuation may be considerably greater than that caused by molecular viscosity. A change in the spectrum of height correlators of the liquid surface is seen in the presence of a coherent wave.

1. Formulation of the Problem. A situation often appears in which a low-frequency oscillation, whose phase may be regarded as determined, is superposed on a turbulent motion. The turbulence results in an additional attenuation of the waves, and due to the modulation of the hydrodynamic motion of the long wave the turbulence becomes inhomogeneous and anisotropic. This interaction between the long (coherent) wave and the turbulence can be taken into consideration in the theory of weak turbulence and this exactly is the content of the present work.

The theory of local isotropic and homogeneous weak turbulence has been developed in the works of Zakharov and Filonenko [1, 2], in which the stationary distribution n_k of quasiparticles, i.e., the normal oscillations of the liquid surface, is obtained, through which the spectral density of the energy of turbulence $E(k) = \omega_k n_k$ is expressed.

$$n_k = c_1 k^{-4} \quad (k < k_0), \qquad n_k = c_2 k^{-17/4} \quad (k > k_0) \tag{1.1}$$

Here the first distribution function corresponds to gravity waves and the second to capillary waves; k_0 is the capillary constant. Due to the dispersion of the waves at the surface of the liquid their interaction may be relatively weak if the mean energy of the agitation is not very large and a wave collapse does not occur. Unlike the spectra of Phillips [3] the distributions (1.1) are obtained for waves without whitecaps. We note that in an incompressible liquid the turbulence is always strong because of the absence of dispersion near vortices that are stationary with respect to the liquid. At the same time for surface waves there is a wide range of disturbance parameters, where the concept of weak gauge of mutually interacting perturbations, which are described by the kinetic equation, is applicable. Waves at the surface of a liquid were investigated by Hasselman [4] with the use of kinetic equations.

In the presence of a long wave the law of dispersion ω_k of short-wave perturbations becomes a function of $\mathbf{r}$ and t. The dependence on $\mathbf{r}$ and t can be obtained if the spectrum of the short-wave perturbations is sought in a system of reference moving with the surface of the liquid executing the long-wave motion. In this system of reference forces of inertia are acting, and due to the inhomogeneous deformation of the surface caused by the long wave, the local scale size changes, while due to the slope of the surface the normal component of the gravity changes. As a result, in the law of dispersion of the short wave there appears a dependence on its position on the long wave. If one changes back to the stationary system of reference,

Khar'kov. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 100-109, January-February, 1973. Original article submitted July 20, 1972.

then in the law of dispersion of short waves a Doppler shift appears along with the corrections associated with the factors mentioned above; this Doppler shift plays an important role. We note that the change of the profile of a short wave on a long wave was investigated in [5], but the Doppler shift is not considered there.

The kinetic equation for the distribution function, proportional to the mean squared modulus of the amplitudes of normal oscillations of the liquid surface, i.e., gravity or capillary waves, can be written also in the spatially inhomogeneous case. It has the form

$$\frac{\partial n_k}{\partial t} + \frac{\partial \omega_k}{\partial \mathbf{k}} \frac{\partial n_k}{\partial \mathbf{r}} - \frac{\partial \omega_k}{\partial \mathbf{r}} \frac{\partial n_k}{\partial \mathbf{k}} + \Gamma_k n_k = I^c(n_k) \tag{1.2}$$

where n_k is the distribution function of the quasiparticles, ω_k is the frequency of the surface oscillations, Γ_k is their attenuation due to molecular viscosity, and $I^c(n_k)$ describes collisional interaction between the quasiparticles appearing due to the nonlinearity of the hydrodynamic equations.

It is evident from Eq. (1.2) that due to the change of the law of dispersion of the quasiparticles the long wave leads to a modulation of the distribution; in the linear (with respect to the amplitude of the long wave) approximation this modulation has an inhomogeneous and anisotropic term $\delta n_k(\mathbf{r}, t)$ proportional to the angles of inclination of the surface caused by the long wave. In turn, due to the interaction of the waves, the modulation of the distribution and the appearance of the dependence on the coordinates lead to the appearance of additional forces acting on the long wave and, as shown below, causing a change in its velocity and an additional attenuation. As seen from (1.2) a force equal to $\partial\omega_k/\partial\mathbf{r}$ acts on a quasiparticle, whereas in the equation for the long wave a reverse force averaged over all quasiparticles appears. Thus a coupled system of equations must appear describing both the turbulence in the presence of the long wave and the long wave in the presence of turbulence. (This system of equations is entirely analogous to the equations describing the propagation of a coherent acoustic wave in quasiparticle–electron systems in metals [6].)

In reality the situation is a little more complicated than that described above, since the coherent wave leads not only to the appearance of means of the form

$$\delta n_k(\mathbf{q}, t) = {}^1/_2 (\langle a_k^* a_{k+q} \rangle + \langle a_{k-q}^* a_k \rangle)$$

where a_k is the amplitude of the normal oscillation and $\mathbf{q}$ is the wave vector of the long wave, but also to the appearance of "anomalous" means of the form $\langle aa^* \rangle$, $\langle a^* a^* \rangle$. Therefore, the complete system of equations also includes equations for these means.

The coupled system of equations describing the interaction of the long coherent wave and turbulence is obtained in Secs. 2 and 3. The additional attenuation of the coherent wave in the presence of turbulence is discussed in Sec. 4. The "collision integrals" describing the processes of interaction of waves in the presence of a long coherent wave are obtained in Sec. 5. The change in the spectrum of the height correlators of the liquid surface, caused by the interaction with the long coherent wave, is investigated in Sec. 6.

2. Derivation of Equations for Coherent Wave in the Presence of Turbulence. The basic system of equations of hydrodynamics for surface agitation has the form

$$\begin{gathered}\left.\frac{\partial\varphi}{\partial t}\right|_{z=\zeta} + g\zeta - \frac{\alpha}{\rho}\Delta\zeta = -\frac{1}{2}(\nabla\varphi)^2|_{z=\zeta} - \frac{\alpha}{2\rho}[\Delta\zeta + (\nabla\zeta\nabla)](\nabla\zeta)^2 \\ \frac{\partial\zeta}{\partial t} - \left.\frac{\partial\varphi}{\partial z}\right|_{z=\zeta} = -(\nabla\zeta\nabla\varphi)|_{z=\zeta}, \quad \Delta\varphi = 0, \quad \varphi|_{z\to-\infty} \to 0\end{gathered} \tag{2.1}$$

where $\varphi(\mathbf{r}, z, t)$ is the velocity potential of the liquid, $\zeta(\mathbf{r}, t)$ is the departure of the surface from the equilibrium position caused by the agitation, the z axis is directed upward, and $z = 0$ corresponds to the surface of the liquid in the absence of agitation.

Following [1, 2] we pass on to canonical variables $\zeta(\mathbf{r}, t)$ and $\psi(\mathbf{r}, t) = \varphi[\mathbf{r}, \zeta(\mathbf{r}, t), t]$ and introduce the normal coordinates of oscillations of the surface a_k and a_k^*

$$\zeta(\mathbf{r}, t) = \int \left(\frac{4k}{g + \alpha k^2/\rho}\right)^{1/4} (a_k e^{i\mathbf{kr}} + a_k^* e^{-i\mathbf{kr}})\, d\mathbf{k} \tag{2.2}$$

$$\psi(\mathbf{r}, t) = -i\int \left[\frac{4(g + \alpha k/\rho)}{k}\right]^{1/4} (a_k e^{i\mathbf{kr}} - a_k^* e^{-i\mathbf{kr}})\, d\mathbf{k}$$

The Hamiltonian of the system of surface waves is

$$H = \frac{\rho}{2}\int d\mathbf{r}\int_{-\infty}^{\zeta}(\nabla\varphi)^2 dz + \rho g\int d\mathbf{r}\int_0^{\zeta} z dz = \int \omega_k a_k{}^* a_k d\mathbf{k} +$$
$$+\int\Big[V^{(1)}_{1,23}a_1{}^*a_2a_3\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)+\frac{1}{3}V^{(2)}_{123}a_1a_2a_3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)+B\Big]\times$$
$$\times\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 + \int\Big[\frac{1}{4}W^{(1)}_{12,34}a_1{}^*a_2{}^*a_3a_4\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)+W^{(2)}_{1,234}a_1{}^*a_2a_3a_4\times$$
$$\times\,\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)+B\Big]d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 \tag{2.3}$$

where for the sake of brevity we have introduced the notation $\mathbf{k}_1 \equiv 1$, $\mathbf{k}_2 \equiv 2$, etc., and B denotes a complex-conjugate quantity. The terms describing the process of generation and annihilation of the four waves are not given, since they do not give any contribution in the approximation considered below. The dynamic equations for the amplitudes a_k and $a_k{}^*$ are obtained by a variation of the Hamiltonian with respect to $ia_k{}^*$ and $-ia_k$:

$$\frac{\partial a_k}{\partial t} = -i\frac{\delta H}{\delta a_k{}^*}, \qquad \frac{\partial a_k{}^*}{\partial t} = i\frac{\delta H}{\delta a_k} \tag{2.4}$$

The presence of the coherent wave leads to the result that the value of the amplitudes of normal oscillations averaged over the ensemble of phases becomes nonzero

$$\langle a_q\rangle = A_q \neq 0 \tag{2.5}$$

Here and below q is the wave vector of the coherent wave. The smallness of the angles of inclination of the coherent wave permits us to restrict ourselves to the linear approximation in respect to A_q. The weak nature of the interaction offers the possibility of investigating the system of surface waves as a set of two systems, coherent and turbulent.

Let us derive the equations of motion for the amplitude of the coherent wave A_q in the presence of turbulence. This can be done by averaging Eq. (2.4) over the statistical ensemble of random phases. Taking (2.5) into consideration, we obtain

$$\frac{\partial A_q}{\partial t} + i\omega_q A_q + 2iA_q\int W^{(1)}_{kq,qk}n_k d\mathbf{k} + 2iA^*_{-q}\int W^{(2)}_{k;k,q,-q}n_k d\mathbf{k} =$$
$$= -i\int\Big[V^{(1)}_{q;k,q-k}\langle a_k a_{q-k}\rangle + 2V^{(1)}_{k+q;q,k}\langle a_k{}^* a_{k+q}\rangle + V^{(2)}_{q,k,k+q}\langle a_k{}^* a^*_{-k-q}\rangle\Big]d\mathbf{k} \tag{2.6}$$

As seen from (2.6) the presence of interaction in the system of surface waves leads to the appearance of the correlation functions $\langle a_k a_{q-k}\rangle$, $\langle a_k{}^* a_{k+q}\rangle$, $\langle a_k{}^* a_{-k-q}\rangle$ in the equation for the amplitude of the coherent wave; these correlation functions determine the additional forces acting on the coherent wave from the side of the turbulent subsystem. These forces appear due to the fact that in the presence of the coherent wave the distribution of the quasiparticles becomes different from (1.1).

3. Description of Turbulence in the Presence of Coherent Wave. In order to obtain a closed system of equations it is necessary to derive the equations for the correlation functions occuring in (2.6). For the correlator $\langle a_k{}^* a_{k+q}\rangle$ we have

$$\Big\{\frac{\partial}{\partial t} + i(\omega_{k+q}-\omega_k)\Big\}\langle a_k{}^* a_{k+q}\rangle = -i\int[V^{(1)}_{k+q,12}\langle a_k{}^* a_1 a_2\rangle\delta(\mathbf{k}+\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2)+$$
$$+2V^{(1)}_{2;k+q,1}\langle a_k{}^* a^*{}_1 a_2\rangle\delta(\mathbf{k}+\mathbf{q}+\mathbf{k}_1-\mathbf{k}_2)+V^{(2)}_{k+q,1,2}\langle a_k{}^* a_1{}^* a_2{}^*\rangle\delta(\mathbf{k}+\mathbf{q}+\mathbf{k}_1+\mathbf{k}_2)-B_0]\,d\mathbf{k}_1 d\mathbf{k}_2 \tag{3.1}$$

Here B_0 denotes complex-conjugate terms with the substitution $\mathbf{k} \rightleftharpoons \mathbf{k}+\mathbf{q}$.

The presence of the coherent wave offers the possibility of expressing the triple correlators in terms of the sum of products of double correlators and the amplitude of the coherent wave $\langle a\rangle$. In the linear (with respect to A_q) approximation the terms that are a product of $\langle a\rangle$ and the anomalous double correlators $\langle a\,a\rangle$ and $\langle a^* a^*\rangle$ must be omitted, since, as will be shown below, the latter are proportional to the amplitude of the coherent wave. In products of the form $\langle a\rangle\langle a^* a\rangle$ only those means are retained where the paired correlator is a stationary function of distribution (1.1). For example, for the correlator $\langle a_k{}^* a_1 a_2\rangle$, we obtain

$$\langle a_k{}^* a_1 a_2\rangle = \langle a_2\rangle n_k\delta(\mathbf{k}-\mathbf{k}_1) + \langle a_1\rangle n_k\delta(\mathbf{k}-\mathbf{k}_2) \tag{3.2}$$

With the above statement taken into consideration Eq. (3.1) reduces to the form

$$\left\{\frac{\partial}{\partial t}+i(\omega_{k+q}-\omega_k)\right\}\langle a_k{}^*a_{k+q}\rangle-2i(n_{k+q}-n_k)\,[V^{(1)}_{k+q;\,k,\,q}A_q+V^{(1)}_{k;\,k+q,\,q}A^*_{-q}]=I_1 \tag{3.3}$$

where I_1 is the linearized collision integral, which differs from the usual integral in the presence of the anomalous correlators. The expression for I_1 is obtained by uncoupling higher correlation functions and is given in Sec. 5.

Below we shall investigate the case where the coherent wave is a long wave ($q \ll k$), and in Eq. (3.3) expansion in $q/k \ll 1$ can be carried out:

$$\left\{\frac{\partial}{\partial t}+i\mathbf{q}\mathbf{v}_k\right\}\langle a_k{}^*a_{k+q}\rangle-2i\mathbf{q}\frac{\partial n_k}{\partial \mathbf{k}}\,[V^{(1)}_{k+q;\,k,\,q}A_q+V^{(1)}_{k;\,k+q,\,q}\overset{*}{A}_{-q}]=I_1 \tag{3.4}$$

where $\mathbf{v}_k = \partial\omega_k/\partial\mathbf{k}$ is the group velocity of the short waves. We note that the representation of the sea surface in the form of long waves, on which there are ripples, has been found useful also in a number of radiophysics problems [7].

As seen from (2.6), along with the normal correlators $\langle a^*a\rangle$ there also appear anomalous correlators $\langle a^*a^*\rangle$ $\langle aa\rangle$, for which the equations are obtained in an analogous way, and for $q \ll k$ they have the form

$$\left\{\frac{\partial}{\partial t}+2i\omega_k\right\}\langle a_k a_{q-k}\rangle+4in_k\,[V^{(1)}_{q;\,k,\,q-k}A_q+V^{(2)}_{k,\,q,\,q-k}\overset{*}{A}_{-q}]=I_2$$
$$\left\{\frac{\partial}{\partial t}-2i\omega_k\right\}\langle a_k{}^*\overset{*}{a}_{-k-q}\rangle-4in_k\,[V^{(2)}_{k,\,q,\,k+q}A_q+V^{(1)}_{q;\,k,\,k+q}\overset{*}{A}_q]=I_3 \tag{3.5}$$

The collision integrals I_2 and I_3 are given below. The equation for the amplitude $A_{-q}{}^*$ and the correlation functions associated with it are obtained from (2.6), (3.4), (3.5) by taking complex conjugates and replacing $\mathbf{q}\rightarrow$ by $-\mathbf{q}$

We note that the force in Eqs. (3.4), (3.5) is proportional to the amplitude of the coherent wave. If with the use of (3.4) in $\mathbf{r}$-representation we write the equation for the correction to the distribution function in the subsystem of quasiparticles $\delta n_k(\mathbf{r}, t)$, caused by the presence of the long coherent wave, then in accordance with (1.2) this equation acquires the sense of the linearized kinetic equation

$$\frac{\partial\delta n_k(r,t)}{\partial t}+(\mathbf{v}_k\nabla)\,\delta n_k(\mathbf{r},t)-\frac{\partial\delta\omega_k(\mathbf{r},t)}{\partial\mathbf{r}}\frac{\partial n_k}{\partial\mathbf{k}}=I^\circ \tag{3.6}$$

where

$$\delta n_k(\mathbf{r},t)=\frac{1}{2}\int[\langle a_k{}^*a_{k+q}\rangle+\langle\overset{*}{a}_{k-q}a_k\rangle]\,e^{i\mathbf{q}\mathbf{r}}d\mathbf{q}$$
$$I^\circ=\frac{1}{2}\int[I_1(\mathbf{q})+I_1{}^*(-\mathbf{q})]\,e^{i\mathbf{q}\mathbf{r}}d\mathbf{q} \tag{3.7}$$

and the correction to the frequency of the short waves due to the modulation of the coherent wave has the form

$$\delta\omega_k(\mathbf{r},t)=2\int[V^{(1)}_{k+q;\,k,\,q}A_q+V^{(1)}_{k;\,k+q,\,q}\overset{*}{A}_{-q}+B_1]\,e^{i\mathbf{q}\mathbf{r}}d\mathbf{q} \tag{3.8}$$

Here B_1 denotes complex-conjugate terms with the replacement $\mathbf{q}\rightarrow -\mathbf{q}$.

The quantity $\delta\omega_k(\mathbf{r}, t)$ includes both the Doppler shift and the corrections associated with the effect of the inertia forces and the change of the local scale size on the long coherent wave; I° differs from I^c in Eq. (1.2) in the presence of anomalous correlation functions.

In the linear approximation in A_q the anomalous correlators appear due to the disruption of the chaotic nature of the phases in the turbulent subsystem in the presence of the coherent wave and due to the presence of terms in the Hamiltonian that describe processes with nonconservation of quasiparticles.

4. Dispersion and Absorption of Long Waves. The equations for the amplitude of the coherent wave A_q (2.6) and the correlation functions $\langle a_k{}^*a_{k+q}\rangle$, $\langle a_k, a_{q-k}\rangle$, $\langle a_k{}^*a_{-k-q}{}^*\rangle$ (3.4), and (3.5), together with the equations for the amplitudes $A_{-q}{}^*$, A_{-q}, $A_q{}^*$ and of the correlation functions associated with them, which are obtained from the above equations by making the substitution $\mathbf{q}\rightarrow -\mathbf{q}$ and by complex conjugation, form a complete system of integrodifferential equations describing the interaction of long coherent

waves and turbulence. Different types of correlation functions are found to be coupled in the equations for the amplitudes of the coherent waves (2.6) as well as in the "collision integrals" (5.1), (5.2). The complete system of equations can be symbolically written in the following way:

$$\hat{D}_{il} A_l = -i \int P_{il} f_l d\mathbf{k} \tag{4.1}$$

$$\hat{K}_{il} f_l + \hat{\nu}_{il} f_l = Q_{il} A_l \tag{4.2}$$

Here A_l is the set of all amplitudes of the coherent waves A_q, A_{-q}^*, A_{-q}, A_q^*; f_i are paired correlation functions (normal and anomalous) induced by the coherent wave with amplitude A_l; the forms of the differential operators $\hat{D}_{il}$, $\hat{K}_{il}$ and matrices P_{il} and Q_{il} are clear from a comparison of (4.1) and (4.2) with Eqs. (2.6), (3.4), (3.5) and the remaining equations occurring in the system. The integral operator describes the interaction in the turbulent subsystem

$$-\hat{\nu}_{1l} f_l = I_1, \quad -\hat{\nu}_{2l} f_l = I_2, \quad -\hat{\nu}_{3l} f_l = I_3$$

As seen from Eq. (2.6), the amplitudes of the coherent waves A_q and A_{-q}^* (and also A_{-q} and A_q^*) are coupled in pairs. However, in the collision integrals all the correlation functions occurring in the equation are coupled with each other; thus the collision integral couples all the four amplitudes A_q, A_{-q}^*, A_{-q}, A_q^*.

In the case of an infinite system the solution of (4.2) in Fourier components with respect to time can be formally written in the form

$$f_i = R_{il}(\omega) Q_{lm} A_m \tag{4.3}$$

where $R_{il}(\omega)$ is the time-Fourier component of the Green operator of equations for the correlation functions. Taking the Fourier transform with respect to time and substituting (4.3) into (4.1), we obtain

$$D_{il}(\omega) A_l = -i\sigma_{im}(\omega) A_m, \quad \sigma_{im} = \int P_{il} R_{ln}(\omega) Q_{nm} d\mathbf{k} \tag{4.4}$$

Separating out the terms connected with the interaction from matrix D_{il}

$$D_{il} = (-\omega + \omega_q^i) \delta_{il} + D_{il}^{\text{int}}, \; \omega_q^i = \pm \omega_q$$

and grouping them with σ_{il}, we rewrite (4.4) in the form

$$\{(-\omega + \omega_q^i) \delta_{il} + d_{il}\} A_l = 0, \quad (d_{il} = D_{il}^{\text{int}} + i\sigma_{il}) \tag{4.5}$$

The expression for d_{il} is not given here; it can be obtained from a comparison with the preceding formulas. Equating the determinant of the homogeneous linear system of equations (4.5) to zero, we obtain the dispersion equation

$$\text{Det} \; | (-\omega + \omega_q^i) \delta_{il} + d_{il} | = 0 \tag{4.6}$$

The analysis of Eqs. (4.4), (4.6) is very complicated. We shall restrict ourselves to the case where the coupling among the different types of correlators in the collision integrals can be disregarded; we write these integrals in τ-approximation:

$$I_1 \sim -\nu_1 \langle a^*_k a_{k+q} \rangle, \quad I_2 \sim -\nu_2 \langle a_k a_{q-k} \rangle, \quad I_3 \sim -\nu_3 \langle a_k^* a^*_{-k-q} \rangle$$

Then equations (4.4) break up into two systems connecting the amplitude A_q with A_{-q}^* and A_{-q} with A_q^*:

$$\begin{gathered} -(\omega - \omega^\circ) A_q + \lambda(\mathbf{q}) A^*_{-q}{}^* = 0 \\ -(\omega + \omega^{\circ *}) A_{-q}^* - \lambda^*(\mathbf{q}) A_q = 0 \end{gathered} \tag{4.7}$$

where

$$\omega^\circ \equiv \omega_q + 2\int \left[W^{(1)}_{kq,\, qk} n_k + \frac{2\,|V^{(1)}_{k+q;\, k,\, q}|^2}{\mathbf{q}\mathbf{v}_k - \omega - i\nu_1} \mathbf{q} \frac{\partial n_k}{\partial \mathbf{k}} - \frac{2\,|V^{(1)}_{q;\, k,\, q-k}|^2 n_k}{2\omega_k - \omega - i\nu_2} - \frac{2\,|V^{(2)}_{k,\, q,\, k+q}|^2 n_k}{2\omega_k + \omega + i\nu_3} \right] d\mathbf{k} \tag{4.8}$$

$$\lambda(q) \equiv \int \left[W^{(2)}_{k;\, q,\, k,\, -q} n_k + \frac{2V^{(1)}_{k+q;\, k,\, q} V^{(1)}_{k-q;\, k,\, q}}{\mathbf{q}\mathbf{v}_k - \omega - i\nu_1} \mathbf{q} \frac{\partial n_k}{\partial \mathbf{k}} - \frac{2V^{(1)}_{q;\, k,\, q-k} V^{(2)}_{k,\, q,\, q-k} n_k}{2\omega_k - \omega - i\nu_2} - \frac{2V^{(1)}_{q;\, k,\, k+q} V^{(2)}_{k,\, q,\, k+q} n_k}{2\omega_k + \omega + i\nu_3} \right] d\mathbf{k}$$

and ω_q is the frequency of the long wave with wave vector q in the absence of turbulence. Similar equations connect A_{-q} and A_q^*. From (4.7) we find that the roots of the dispersion equation are

$$\omega_{1,2} = \pm\sqrt{|\omega_q^\circ|^2 - |\lambda(q)|^2 - (\mathrm{Im}\,\omega_q^\circ)^2} + i\,\mathrm{Im}\,\omega_q^\circ \tag{4.9}$$

It is evident from (4.7) that in the linear approximation in respect of the interaction the roots of the dispersion equation have the form

$$\omega_1 = \omega_q^\circ, \qquad \omega_2 = -\omega_q^{\circ *}$$

It is seen from (4.1) that because of the interaction through the turbulent subsystem waves with wave vectors **q** and **−q** get coupled. As a result, the frequency degeneracy in long waves propagating in opposite directions is removed and new types of oscillations appear, whose normal coordinates b_q and b_{-q}^* have the following form in the linear approximation in respect of $\lambda(q)/\omega_q$:

$$b_q = A_q + \frac{\lambda(\mathbf{q})}{2\omega_q}A_{-q}^*, \qquad b_{-q}^* = A_{-q}^* + \frac{\lambda^*(\mathbf{q})}{2\omega_q}A_q \tag{4.10}$$

Thus the propagation of a coherent wave in a region with statistically homogeneous isotropic turbulence is accompanied by the appearance of a reflected wave with reflection coefficient $\lambda(q)/2\omega_q$. Making use of the estimate for the matrix elements for $q \ll k$

$$V^{(1)}_{k+q;\,k,\,q} \sim -V^{(1)}_{k;\,k+q,\,q} \sim g^{1/4}kq^{3/4}\cos(\hat{\mathbf{k},\mathbf{q}}), \qquad V^{(2)}_{k,\,q,\,k+q} \sim -V^{(1)}_{q;\,k,\,q-k} \sim g^{1/4}kq^{3/4}$$

and of the collision frequencies $\nu(k) \sim c_1^2k^2\omega_k^{-1}$ [1, 2] for gravity ripples and $\nu(k) \sim c_2k^{3/4}$ for capillary ripples (see also Sec. 5), we find that the reflection coefficient of a long gravity wave in the presence of gravity ripples is of the order

$$\frac{\lambda(\mathbf{q})}{2\omega_q} \sim \sqrt{\frac{\nu(k_a)}{\omega_a}\frac{q}{k_a}} + \frac{i\omega_q}{\sqrt{\nu(k_a)\,\omega_a}}\left(\frac{q}{k_a}\right)^{3/2}$$

In the presence of capillary ripples the reflection coefficient is of the form

$$\frac{\lambda(\mathbf{q})}{2\omega_q} \sim \frac{\nu(k_a)}{\omega_a}\frac{q}{k_a} - \frac{\omega_q}{\nu(k_a)}\left(\frac{q}{k_a}\right)^2 + i\left(\frac{q}{k_a}\right)^2$$

Here ω_a is the frequency corresponding to the low-frequency boundary of the inertial range of gravity wave turbulence and $\mathbf{k}_a$ is the low-frequency boundary of the range of gravity or capillary turbulence.

Let us consider the additional attenuation of the coherent wave in the presence of turbulence. The long coherent wave causes a departure of the distribution of the quasiparticles from the stationary distribution (1.1). The interaction in the turbulent subsystem restores the stationary distribution and thereby leads to additional attenuation of the coherent wave.

Evaluating (4.8) under the condition $\nu^{-1}(\mathbf{k}) \ll 1$, we find that when a long gravity wave propagates in a region with gravity ripples, the decrement of the additional attenuation of the long wave is in order of magnitude equal to

$$\frac{\mathrm{Im}\,\omega}{\omega} \sim \frac{\omega_q}{\sqrt{\omega_a\nu(k_a)}}\left(\frac{q}{k_a}\right)^{3/2} \tag{4.11}$$

In the case of capillary ripples the decrement of attenuation has the form

$$\mathrm{Im}\,\omega/\omega \sim (q/k_a)^2 \tag{4.12}$$

The additional attenuation of a long gravity wave prevails over the viscous attenuation for $c_1 \ll (q/k_a)^{1/2}\omega/\gamma k_a^2$ in the presence of gravity ripples and for $q \gg (\gamma k_a)g^{-1}$ in the presence of capillary ripples, where γ is kinematic viscosity of water. The attenuation of long waves in turbulence will be investigated in greater detail separately.

5. Collision Integral. The collision integrals I_1, I_2, I_3 in Eqs. (3.4), (3.5), describing the processes of interaction of waves in the turbulent subsystem in the presence of a coherent wave, are obtained by decoupling the chains of equations for the correlation functions (see [8]). It is convenient to make use of notation similar to those used by Kasselman [4] for writing the unwieldy expressions thus obtained; we introduce an additional parameter s which takes the values ±1, wherein

$$a^s = \begin{cases} a^+ \equiv a^*, & s = +1 \\ a^- \equiv a, & s = -1 \end{cases}$$

With this notation Eqs. (2.6), (3.4), (3.5) become

$$\left\{\frac{\partial}{\partial t} - is\omega_q\right\} A_{sq}{}^s = -is \sum_{s's''} \int V_{q,\,k,\,q-k}^{-ss's''} \langle a_{s'k}^{s'} a_{s''(q-k)}^{s''} \rangle \, d\mathbf{k} \tag{5.1}$$

$$\left\{\frac{\partial}{\partial t} - i\,(s'\omega_k + s''\omega_{q-k})\right\} \langle a_{s'k}^{s'} a_{s''(q-k)}^{s''} \rangle + 2i\,(s'n_k + s''n_{q-k}) \sum_{s_1} V_{k,\,q-k,\,q}^{-s',-s'',\,s_1} A_{s_1q}^{s_1} = I^{s's''}$$

where $I^{s's''}$ are collision integrals

$$I^{+-}(-\mathbf{q}) \equiv I_1, \quad I^{--}(-\mathbf{q}) \equiv I_2, \quad I^{++}(-\mathbf{q}) \equiv I_3$$

In the case of capillary (decay) turbulence for q ≪ k the collision integrals are

$$I^{s's''} = -2\pi \sum_{s_1s_2s_3} \int \{s'V_{k12}^{-s's_1s_2} [V_{k12}^{s_3,-s_1,-s_2} (s_1n_2 + s_2n_1) \langle a_{s_3}{}^{s_3} a_{s''(q-k)}^{s''} \rangle \times$$
$$\times\, \delta(s_3\omega_k - s_1\omega_1 - s_2\omega_2) + 2V_{21k}^{s_3,-s_1,-s''} (s_1n_k + s''n_1) \times$$
$$\times \langle a_{s_3(q-k_2)}^{s_3} a_{s_2k_2}^{s_2} \rangle \, \delta(s_3\omega_2 - s_1\omega_1 - s''\omega_k)] \, \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) + B_2\} \, d\mathbf{k}_1 d\mathbf{k}_2 \tag{5.2}$$

Here B_2 denotes terms with the substitution s' ⇌ s" and simultaneous substitution **k** ⇌ **q** − **k**.

In order to obtain collision integrals in the case of gravity (nondecay) turbulence we first carry out a canonical transformation with Hamiltonian (2.3) so as to eliminate terms that are cubic in the normal coordinates. It is necessary to keep in mind that in the presence of a coherent wave the value of the normal coordinates averaged over the ensemble of phases is nonzero. Therefore, before carrying out the canonical transformation we must express the normal coordinates in the form $a^s = a'^s + \langle a^s \rangle$, after which of the cubic terms in the Hamiltonian we separate out those that are linear in $\langle a \rangle$. The separated terms, together with

$$H_0 = \int \omega_k a_k{}^* a_k d\mathbf{k}$$

will be the new (unnormalized due to the presence of the coherent wave) Hamiltonian of noninteracting quasiparticles

$$\widetilde{H}_0 = H_0 + \sum_{s_1s_2s_3} \int V_{123}^{s_1s_2s_3} a_1{}^{s_1} a_2{}^{s_2} \langle a_3{}^{s_3} \rangle \, \delta(s_1\mathbf{k}_1 + s_2\mathbf{k}_2 + s_3\mathbf{k}_3) \, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

after which the canonical transformation can be carried out, but now with H_0. The separation of the coherent wave in the normal coordinates makes it possible to take into consideration the renormalization of the frequency in the Hamiltonian in the linear approximation in respect of A and results in the appearance of the force term in the kinetic equation. For gravity turbulence the collision integrals for q ≪ k have the form

$$I_g^{s's''} = -\frac{2\pi}{3} \sum_{s_1s_2s_3s_4} \int \{s'T_{k123}^{-s's_1s_2s_3} [T_{k123}^{s_4,-s_1,-s_2,-s_3} (s_1n_2n_3 + s_2n_1n_3 +$$
$$+ s_3n_1n_2) \langle a_{s''(q-k)}^{s''} a_{s_4k}^{s_4} \rangle \, \delta\,(s_4\omega_k - s_1\omega_1 - s_2\omega_2 - s_3\omega_3) + 3T_{1k23}^{s_4,-s'',-s_2,-s_3} \times$$
$$\times (s''n_2n_3 + s_2n_kn_3 + s_3n_kn_2) \langle a_{s_1k_1}^{s_1} a_{s_4(q-k_1)}^{s_4} \rangle \, \delta(s_4\omega_1 - s''\omega_k - s_2\omega_2 - s_3\omega_3)] \times$$
$$\times\, \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) + B_3\} \, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \tag{5.3}$$

Here B_3 denotes terms with the replacement s' ⇌ s" and simultaneously **k** ⇌ **q** − **k**; T is the matrix element of the "effective" four-partial Hamiltonian (see [2]), in which the third-order terms in a' have been eliminated with the canonical transformation.

6. Spectrum of Turbulence in the Presence of Long Waves. The Fourier component of the height correlator of the liquid surface is given by the relation

$$S(\mathbf{k}, \mathbf{r}, t) = \frac{1}{(2\pi)^2} \int \langle \zeta(\mathbf{r}, t)\, \zeta(\mathbf{r} + \boldsymbol{\rho}, t) \rangle \, e^{-i\mathbf{k}\boldsymbol{\rho}} d\boldsymbol{\rho} \tag{6.1}$$

In the stationary case, the following simple relation exists between the distribution function n_k and the spectrum S(k):

$$S(k) = \left(\frac{4k}{g + \alpha k^2/\rho}\right)^{1/2} n_k$$

Making use of the formulas for changing over to canonical variables (2.2) we find that the presence of the long coherent wave leads to a modulation of the spectrum and to the appearance of an additional term in the spectrum

$$S(\mathbf{k}, \mathbf{r}, t) = S(k) + \delta S(\mathbf{k}, \mathbf{r}, t)$$

depending on the coordinates and time:

$$\delta S(\mathbf{k}, \mathbf{r}, t) = \left(\frac{4k}{g + \alpha\rho^{-1}k^2}\right)^{1/2} e^{i\mathbf{qr}} \{\langle a_k a_{q-k}\rangle + \langle a^*_{-k} a^*_{k-q}\rangle + \langle a^*_{-k} a_{q-k}\rangle + \langle a^*_{k-q} a_k\rangle\} \tag{6.2}$$

Substituting the expressions for the correlators from (4.3), we find that in the linear (in respect of the amplitude of the coherent wave) approximation the inhomogeneous part of the spectrum of the turbulence, caused by the presence of the inhomogeneous wave, is of the form

$$\delta S(\mathbf{k}, \mathbf{r}, t) = \left(\frac{4k}{g + \alpha\rho^{-1}k^2}\right)^{1/2} e^{i\mathbf{qr}} \sum_j R_{jl}(\omega)\, Q_{lm} A_m \tag{6.3}$$

where the index j takes the values from 1 to 4 corresponding to four correlators in (6.2). For an estimate of δS(**k**, **r**, t) we replace the collision integrals by the relaxation times as before; then (6.3) becomes

$$\delta S(\mathbf{k}, \mathbf{r}, t) = -\left(\frac{4k}{g + \alpha\rho^{-1}k}\right)^{1/2} e^{i\mathbf{qr}} \left\{\frac{4n_k (V^{(1)}_{q;\,k,\,q-k} A_q + V^{(2)}_{k,\,q,\,q-k} A^*_{-q})}{2\omega_k - \omega - i\nu} - 2\mathbf{q}\frac{\partial n_k}{\partial \mathbf{k}} \frac{V^{(1)}_{q-k;\,k,\,q} A_q + V^{(1)}_{k;\,q-k,\,q} A^*_{-q}}{-\omega + \mathbf{qv}_k - i\nu} + B_4\right\} \tag{6.4}$$

Here B_4 denotes complex conjugate terms with the replacements $\mathbf{k} \to -\mathbf{k}$, $\mathbf{q} \to -\mathbf{q}$, $\omega \to -\omega$.

Making an order-of-magnitude estimate of (6.4) for $q \ll k_0^2/k$, $\omega\nu^{-1}(k) \ll 1$, we have

$$\operatorname{Re}\delta S(\mathbf{k}, \mathbf{r}, t) \sim q\zeta^\circ(\mathbf{r}, t)\left[\left(\frac{\omega^q}{\omega_k}\right)^2 \frac{k}{q} + \frac{2\omega_q^2}{\nu^2}\cos^2(\hat{\mathbf{k}, \mathbf{q}})\right] S(k) \tag{6.5}$$

where $\zeta^\circ(\mathbf{r}, t)$ is the height of the long wave.

For the case of gravity turbulence, considering the smallness of the quantity $[\omega_q/\nu(k)]^2 \ll 1$, we have

$$\delta S(\mathbf{k}, \mathbf{r}, t) \sim q\zeta^\circ S(k)$$

For capillary turbulence we have

$$\text{for } \left(\frac{k_0}{k}\right)^2 \gg \left[\frac{\omega_q}{\nu(k)}\right]^2 \quad \delta S(\mathbf{k}, \mathbf{r}, t) \sim q\zeta^\circ(\mathbf{r}, t)\left[\frac{\omega_q}{\nu(k)}\right]^2 S(k)\cos(\hat{\mathbf{k}, \mathbf{q}})$$

The last result can be obtained if we keep in mind that in the case under investigation we can write

$$\operatorname{Re}\frac{\delta S(\mathbf{k}, \mathbf{r}, t)}{S(\mathbf{k})} \sim \operatorname{Re}\frac{\delta n_k(\mathbf{r}, t)}{n_k}$$

For $qv_k \ll \omega_q$, $\omega_q \nu^{-1}(k) \ll 1$, from the kinetic equation (3.6) we get

$$\operatorname{Re}\delta n_k(\mathbf{r}, t) \sim \delta\omega_k(\mathbf{r}, t)\, \mathbf{q}\frac{\partial n_k}{\partial \mathbf{k}} \frac{\omega_q}{\nu^2(k)} \tag{6.6}$$

The additional term in the law of dispersion of short waves is mainly due to the Doppler shift of the frequency. Thus

$$\delta\omega_k \sim \mathbf{ku}(\mathbf{r}, t) \sim \omega_k \sqrt{kq}\, \zeta^\circ \cos(\hat{\mathbf{k}, \mathbf{q}})$$

where **u**(**r**, t) is the rate of displacement of the liquid surface due to the presence of the long gravity wave. Substituting the last expression into (6.6) we get

$$\operatorname{Re}\frac{\delta S(\mathbf{k}, \mathbf{r}, t)}{S(k)} \sim q\zeta^\circ(\mathbf{r}, t)\left[\frac{\omega_q}{\nu(k)}\right]^2 \cos(\hat{\mathbf{k}, \mathbf{q}})$$

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